Analysis of a Thin Circular Loop Antenna Over a Homogeneous Earth

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(Manuscript received March 31, 1969)

In this paper, the current distribution on a bare conducting loop, situated in free space over a semi-infinite medium, is obtained for arbitrary time harmonic excitations. The loop is assumed to be thin, perfectly conducting and the standard one-dimensional integral equation and its Fourier series solution are used as the starting points. The field due to the current in the loop, where the semi-infinite medium is absent, is expressed as a superposition of plane waves. The tangential component of the field reflected by the interface, of the semi-infinite medium, is evaluated using appropriate Fresnel reflection coefficients. This reflected field serves as a new source for the loop and induces a current on the loop. The field due to the induced current is treated in the same manner, and this process is repeated indefinitely. The summation of the original current and all the induced currents gives the steady-state current on the loop.

I. INTRODUCTION

It is well known that a high altitude nuclear burst generates an intense electromagnetic transient which covers a large geographical area.[1] This transient field induces currents in communication circuits and, if these are large enough, adversely affects communication channels. One problem of particular interest in land-line communication is the coupling to large loops formed by cables.

The loops formed by cables deployed in practical communication systems are very complex and cannot be analyzed easily. Typically, they run for many miles over inhomogeneous terrain and contain many junction points; nevertheless a great deal of insight into the behavior of these irregular loops can be obtained by studying the behavior of a large regular loop over a homogeneous ground. In this paper, the theoretical foundations for an analysis of a circular loop over a homogeneous ground are developed, for time harmonic excitations. The

response of the loop for transient fields may be obtained by standard Fourier transform techniques.

Various problems related to the thin circular loop have been considered by numerous authors. These may be broadly classified under two categories: loop in an infinite homogeneous medium and loop in a stratified medium.

In the first category, Pocklington, Oseen, Hallen, Storer, and Wu have analyzed the problem of a bare thin perfectly conducting circular loop in free space.[2–6] All these authors use the Fourier series expansion to solve tbe integral equation for the current in the loop. A good analysis of the problem is given in Wu's paper. Adachi and Mushiake analyze the same problem by solving the integrodifferential equation for the current using an iterative method.[7,8] Mei, Baghdasarian and Angelakos, and Tang have discussed the direct numerical solution of the integral equation.[9–11] A variational approach for determining the scattering cross section of a loop is given by Kouyoumjian.[12] Problems concerning loaded loops are considered by Iizuka, Harrington and Ryerson, and Harrington and Mautz.[13–15] The analysis of a loop in a conducting medium is an extension of the analysis of a loop in free space and lends itself to certain approximations. Kraichman, Cben and King and King, and Harrison and Tingley have discussed the bare loop in a dissipative medium;[16–18] Galejs has discussed an insulated loop in a dissipative medium.[19] Finally, the solution to the problem of two identical coaxial coupled loops in a homogeneous medium has been solved by Iizuka, King and Harrison.[20]

In the second category the literature is mostly on small loops or magnetic dipoles over different types of media and is very extensive. (See Ref. 21 for an extensive bibliography.) Wait has considered the problem of loops over a homogeneous earth;[22,23] recently, Sinha and Bhattacharya have analyzed the problem of a vertical magnetic dipole buried inside tbe earth.[24]

The treatment of a small current carrying loop as a magnetic dipole, while satisfactory for many purposes, is nevertheless inexact. Moreover, we do encounter situations where the loop diameter is comparable to the wavelength of the excitation frequency and here we cannot assume the current to be uniformly distributed on the loop. A typical example would be the excitation of a loop by a narrow electromagnetic pulse which contains a broad spectrum of frequencies. The purpose of this paper is to solve the problem of a bare loop over a homogeneous earth taking into account the current distribution on the loop.

Specifically, the system under consideration is a bare, thin, perfectly

conducting circular loop of mean radius b, formed by bending a cylindrical wire of radius a, situated in free space with its plane parallel to and at a distance d from the interface of a semi-infinite, linear homogeneous, isotropic medium (Fig. 1). The loop is excited by an electromagnetic wave of harmonic ($\exp j\omega t$) time variation (the slice generator used to compute the admittance being a limiting case). It is assumed that $k_o a \ll 1$ and $a \ll b$, so that, if the loop were situated in a homogeneous medium the current distribution induced by a specified time harmonic excitation is given by the so-called one-dimensional integral equation.[6] In addition it is assumed that $d \gg a$.

It is desired to determine the current distribution $I(\phi)$ on the loop in the aforementioned system. This is accomplished in the following three, more or less self-contained, sections. In Section II the field of a circular filamentary current in free space is expressed as a superposition of plane waves. Section III evaluates the reflected field when an arbitrary field, of the general form obtained in Section II, is incident on the interface between free space and the semi-infinite medium. In Section IV the results of Sections II and III are combined with the integral equation for the current on a loop in free space to determine the steady state current $I(\phi)$ on the loop using a recurrent "reflection-induction" scheme.

II. THE ELECTROMAGNETIC FIELD OF A CIRCULAR FILAMENTARY CURRENT

Consider a circular filament of current $I(\phi)$ (Fig. 2a) of radius b situated in free space. The coordinate system is so chosen that the loop is parallel to the xy plane at a distance d from it. The loop current may be expressed as a surface current density $\mathbf{K}$, in the $z = d$ plane, in the following manner.

$$\mathbf{K} = \mathbf{a}_\phi I(\phi)\delta(\rho - b)\delta(z - d) \tag{1}$$

where ρ, ϕ and z are the cylindrical coordinates and $\mathbf{a}_\phi$ the unit vector in the ϕ direction. The electromagnetic field due to $\mathbf{K}$ may be expressed as a superposition of plane waves as follows.[25]

$$\mathbf{H}^{(i)} = \iint_{-\infty}^{+\infty} [\pm P, \pm Q, (lP + mQ)(1 - l^2 - m^2)^{-\frac{1}{2}}] \cdot \exp\{jk_o[lx + my \mp (1 - l^2 - m^2)^{\frac{1}{2}}(z - d)]\}\, dl\, dm, \qquad z \gtrless d. \tag{2}$$

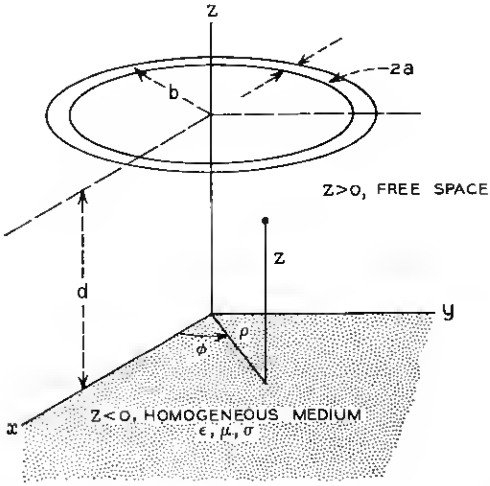

Fig. 1—A thin circular loop over a homogeneous semi-infinite medium.

$$\mathbf{E}^{(i)} = \eta_o \iint_{-\infty}^{\infty} \{[lmP + (1 - l^2)Q](1 - l^2 - m^2)^{-\frac{1}{2}} - [(1 - m^2)P + lmQ](1 - l^2 - m^2)^{-\frac{1}{2}}, \pm(lQ - mP)\} \cdot \exp\{jk_o[lx + my \mp (1 - l^2 - m^2)^{\frac{1}{2}}(z - d)]\}\, dl\, dm, \quad z \gtrless d, \tag{3}$$

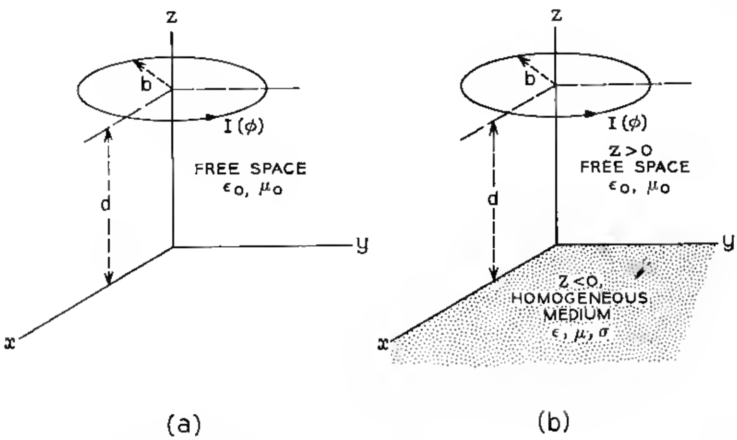

Fig. 2—(a) A filamentary loop current in free space, (b) A filamentary loop current over a homogeneous semi-infinite medium.

where

$$P(l, m) = (k_o^2/8\pi^2) \iint_{-\infty}^{+\infty} K_y(x, y) \exp[-jk_o(lx + my)]\, dx\, dy, \tag{4}$$

$$Q(l, m) = -(k_o^2/8\pi^2) \iint_{-\infty}^{+\infty} K_x(x, y) \exp[-jk_o(lx + my)]\, dx\, dy, \tag{5}$$

$$k_o^2 = \omega^2\mu_o\epsilon_o \quad \text{and} \quad \eta_o = (\mu_o/\epsilon_o)^{\frac{1}{2}}. \tag{6}$$

The quantities l and m are in general complex and the integrals in equations (2) and (3) are contour integrals, the choice of contours being dictated by physical considerations. A possible choice of l and m is defined by the following transformation

$$\left.\begin{aligned} l &= \tau \cos \psi, \\ m &= \tau \sin \psi, \end{aligned}\right\} \quad 0 \leqq \tau, \quad \psi \text{ being complex.} \tag{7}$$

Let

$$I(\phi) = \sum_{n=-\infty}^{+\infty} I_n \exp(jn\phi). \tag{8}$$

Substitution of equations (1), (7) and (8) into equations (4) and (5), and changing from rectangular to cylindrical coordinates yields the following equations

$$P(\tau, \psi) = (k_o^2 b/8\pi) \sum_{n=-\infty}^{+\infty} (-j)^n \exp(jn\psi) J_n(k_o b\tau)(I_{n-1} + I_{n+1}), \tag{9}$$

$$Q(\tau, \psi) = -j(k_o^2 b/8\pi) \sum_{n=-\infty}^{+\infty} (-j)^n \exp(jn\psi) J_n(k_o b\tau)(I_{n-1} - I_{n+1}), \tag{10}$$

where J_n denotes the Bessel function of order n.

III. CIRCULAR FILAMENTARY CURRENT OVER A SEMI-INFINITE MEDIUM

Here again we consider the filamentary loop of Section II, but instead of being situated in free space it is situated over the homogeneous semi-infinite medium $z \leq 0$ (Fig. 2b). The total field in the region $z \geqq 0$ consists of the primary field $\mathbf{E}^{(i)}$, $\mathbf{H}^{(i)}$ of the filamentary current [equations (2) and (3)] and the field $\mathbf{E}^{(r)}$, $\mathbf{H}^{(r)}$ reflected by the interface $z = 0$. We proceed as follows to evaluate the latter. Let

$$\mathbf{H}^{(i)}(x, y, z) = \iint_{-\infty}^{+\infty} \mathbf{H}_o(l, m) \exp[-jk_o\mathbf{n}_o(l, m)\cdot\mathbf{r}]\, dl\, dm, \qquad z \leqq d, \tag{11}$$

$$\mathbf{E}^{(i)}(x, y, z) = \iint_{-\infty}^{+\infty} \mathbf{E}_o(l, m) \exp[-jk_o\mathbf{n}_o(l, m)\cdot\mathbf{r}]\, dl\, dm, \qquad z \leqq d, \tag{12}$$

where

$$\mathbf{H}_o(l, m) = [-\mathbf{a}_x P - \mathbf{a}_y Q + \mathbf{a}_z(lP + mQ)(1 - l^2 - m^2)^{-\frac{1}{2}}] \cdot \exp[-jk_o d(1 - l^2 - m^2)^{\frac{1}{2}}], \tag{13}$$

$$\mathbf{E}_o(l, m) = \eta_o\{\mathbf{a}_x[lmP + (1 - l^2)Q](1 - l^2 - m^2)^{-\frac{1}{2}} + \mathbf{a}_y[(m^2 - 1)P - lmQ](1 - l^2 - m^2)^{-\frac{1}{2}} + \mathbf{a}_z(mP - lQ)\} \cdot \exp[-jk_o d(1 - l^2 - m^2)^{\frac{1}{2}}], \tag{14}$$

$$\mathbf{n}_o(l, m) = -\mathbf{a}_x l - \mathbf{a}_y m - \mathbf{a}_z(1 - l^2 - m^2)^{\frac{1}{2}}, \tag{15}$$

and

$$\mathbf{r} = \mathbf{a}_x x + \mathbf{a}_y y + \mathbf{a}_z z. \tag{16}$$

Each one of the constituent plane waves propagates in the "direction" $\mathbf{n}_o(l, m)$. Let $R_\perp(l, m)$ and $R_\parallel(l, m)$ represent the Fresnel reflection coefficients for the cases where the incident electric field is perpendicular and parallel respectively to the plane of incidence. Evaluation of the reflected field is achieved by resolving each plane wave into components with the electric field perpendicular and parallel to the plane of incidence. To this end we define a local coordinate system† as shown in Fig. 3.

The plane of incidence is defined by the unit vectors $\mathbf{a}_z$ and $\mathbf{n}_o(l, m)$. Let

$$\begin{aligned} \mathbf{a}_1 &= (\mathbf{a}_z \times \mathbf{n}_o)/[1 - (\mathbf{a}_z\cdot\mathbf{n}_o)^2]^{\frac{1}{2}}, \\ \mathbf{a}_2 &= \mathbf{a}_z \times \mathbf{a}_1 . \end{aligned} \tag{17}$$

Then $\mathbf{a}_z$, $\mathbf{a}_1$ and $\mathbf{a}_2$ form a right-handed coordinate system, $\mathbf{a}_1$ is normal to the plane of incidence and $\mathbf{a}_z$ and $\mathbf{a}_2$ lie in the plane of incidence. In terms of $\mathbf{a}_x$ and $\mathbf{a}_y$ we have

$$\mathbf{a}_1 = (l^2 + m^2)^{-\frac{1}{2}}(m\mathbf{a}_x - l\mathbf{a}_y), \tag{18}$$

$$\mathbf{a}_2 = (l^2 + m^2)^{-\frac{1}{2}}(l\mathbf{a}_x + m\mathbf{a}_y). \tag{19}$$

† The propagation vector $\mathbf{n}_0$ becomes complex for certain values of l and m, the associated plane wave being inhomogenous. When this happens some of the terms used in the analysis (for example, plane of incidence, coordinate system, normal, and so on) become inaccurate and should be interpreted in a generalized sense. The results obtained are quite general and valid.

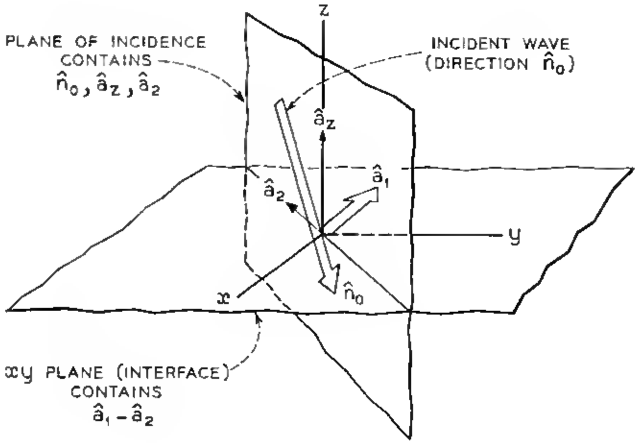

Fig. 3—Local coordinate system.

The field vectors $\mathbf{H}_o$ and $\mathbf{E}_o$ are now resolved into components perpendicular ($\mathbf{H}_{o\perp}$, $\mathbf{E}_{o\perp}$) and parallel ($\mathbf{H}_{o\parallel}$, $\mathbf{E}_{o\parallel}$) to the plane of incidence as follows

$$\mathbf{H}_o = \mathbf{H}_{o\perp} + \mathbf{H}_{o\parallel} , \tag{20}$$

where

$$\mathbf{H}_{o\perp} = \mathbf{a}_1(l^2 + m^2)^{-\frac{1}{2}}(lQ - mP) \exp\left[-jk_o d(1 - l^2 - m^2)^{\frac{1}{2}}\right], \tag{21}$$

$$\begin{aligned}\mathbf{H}_{o\parallel} = [\mathbf{a}_z(1 - l^2 - m^2)^{-\frac{1}{2}} - \mathbf{a}_2(l^2 + m^2)^{-\frac{1}{2}}](lP + mQ)& \\ \cdot\exp\left[-jk_o d(1 - l^2 - m^2)^{\frac{1}{2}}\right],&\end{aligned} \tag{22}$$

$$\mathbf{E}_o = \mathbf{E}_{o\perp} + \mathbf{E}_{o\parallel} , \tag{23}$$

where

$$\begin{aligned}\mathbf{E}_{o\perp} = \mathbf{a}_1\eta_o(l^2 + m^2)^{-\frac{1}{2}}(1 - l^2 - m^2)^{-\frac{1}{2}}(lP + mQ)& \\ \exp\left[-jk_o d(1 - l^2 - m^2)^{\frac{1}{2}}\right],&\end{aligned} \tag{24}$$

$$\begin{aligned}\mathbf{E}_{o\parallel} = \eta_o[\mathbf{a}_z - \mathbf{a}_2(l^2 + m^2)^{-\frac{1}{2}}(1 - l^2 - m^2)^{\frac{1}{2}}](mP - lQ)& \\ \cdot\exp\left[-jk_o d(1 - l^2 - m^2)^{\frac{1}{2}}\right].&\end{aligned} \tag{25}$$

It may be easily verified that

$$\mathbf{H}_{o\parallel} = \eta_o^{-1}(\mathbf{n}_o \times \mathbf{E}_{o\perp}), \tag{26}$$

$$\mathbf{E}_{o\parallel} = -\eta_o(\mathbf{n}_o \times \mathbf{H}_{o\perp}). \tag{27}$$

The reflected field corresponding to the incident field defined in equations (11) and (12) may be represented as

$$\mathbf{H}^{(r)}(x, y, z) = \iint_{-\infty}^{+\infty} \mathbf{H}_2(l, m) \exp\left[-jk_o \mathbf{n}_2(l, m)\cdot\mathbf{r}\right] dl\, dm, \qquad z \geqq 0, \tag{28}$$

$$\mathbf{E}^{(r)}(x, y, z) = \iint_{-\infty}^{+\infty} \mathbf{E}_2(l, m) \exp\left[-jk_o \mathbf{n}_2(l, m)\cdot\mathbf{r}\right] dl\, dm, \qquad z \geqq 0, \tag{29}$$

where

$$\mathbf{n}_2(l, m) = -\mathbf{a}_x l - \mathbf{a}_y m + (1 - l^2 - m^2)^{\frac{1}{2}} \mathbf{a}_z . \tag{30}$$

Let

$$\mathbf{H}_2 = \mathbf{H}_{2\perp} + \mathbf{H}_{2\parallel} , \tag{31}$$

$$\mathbf{E}_2 = \mathbf{E}_{2\perp} + \mathbf{E}_{2\parallel} . \tag{32}$$

According to Fresnel's laws

$$\mathbf{E}_{2\perp} = R_\perp(l, m)\mathbf{E}_{o\perp} , \tag{33}$$

$$\mathbf{H}_{2\perp} = R_\parallel(l, m)\mathbf{H}_{o\perp} , \tag{34}$$

where

$$R_\perp(l, m) = \{\mu k_o(1 - l^2 - m^2)^{\frac{1}{2}} - \mu_o[k^2 - k_o^2(l^2 + m^2)]^{\frac{1}{2}}\}/ \{\mu k_o(1 - l^2 - m^2)^{\frac{1}{2}} + \mu_o[k^2 - k_o^2(l^2 + m^2)]^{\frac{1}{2}}\}, \tag{35}$$

$$R_\parallel(l, m) = \{\mu_o k^2(1 - l^2 - m^2)^{\frac{1}{2}} - \mu k_o[k^2 - k_o^2(l^2 + m^2)]^{\frac{1}{2}}\}/ \{\mu_o k^2(1 - l^2 - m^2)^{\frac{1}{2}} + \mu k_o[k^2 - k_o^2(l^2 + m^2)]^{\frac{1}{2}}\}, \tag{36}$$

$$k^2 = -j\omega\mu(\sigma + j\omega\epsilon). \tag{37}$$

We also have the relations

$$\mathbf{H}_{2\parallel} = \eta_o^{-1}(\mathbf{n}_2 \times \mathbf{E}_{2\perp}), \tag{38}$$

$$\mathbf{E}_{2\parallel} = -\eta_o(\mathbf{n}_2 \times \mathbf{H}_{2\perp}). \tag{39}$$

Substitution of equations (33), (34), (38) and (39) into equations (31) and (32) yields

$$\mathbf{E}_2 = R_\perp \mathbf{E}_{o\perp} - \eta_o R_\parallel(\mathbf{n}_2 \times \mathbf{H}_{o\perp}), \tag{40}$$

$$\mathbf{H}_2 = R_\parallel \mathbf{H}_{o\perp} + \eta_o^{-1} R_\perp(\mathbf{n}_2 \times \mathbf{E}_{o\perp}). \tag{41}$$

Equations (40), (41), (28) and (29) specify the reflected field completely. The ϕ component of the electric field is of special interest since it induces a current in an actual circular loop. Taking the ϕ component of equation (40), using polar coordinates for x, y, and equation (7), we obtain

$$\begin{aligned} E_{2\phi} = \eta_o[R_\parallel(\tau)(1 - \tau^2)^{\frac{1}{2}}(P \sin \psi - Q \cos \psi) \sin (\psi - \phi) \\ - R_\perp(\tau)(1 - \tau^2)^{-\frac{1}{2}}(P \cos \psi + Q \sin \psi) \cos (\psi - \phi)] \\ \cdot \exp [-jk_o d(1 - \tau^2)^{\frac{1}{2}}], \end{aligned} \tag{42}$$

where

$$\begin{aligned} R_\parallel(\tau) = [\mu_o k^2(1 - \tau^2)^{\frac{1}{2}} - \mu k_o(k^2 - k_o^2\tau^2)^{\frac{1}{2}}]/ \\ [\mu_o k^2(1 - \tau^2)^{\frac{1}{2}} + \mu k_o(k^2 - k_o^2\tau^2)^{\frac{1}{2}}], \end{aligned} \tag{43}$$

$$\begin{aligned} R_\perp(\tau) = [\mu k_o(1 - \tau^2)^{\frac{1}{2}} - \mu_o(k^2 - k_o^2\tau^2)^{\frac{1}{2}}]/ \\ [\mu k_o(1 - \tau^2)^{\frac{1}{2}} + \mu_0(k^2 - k_o^2\tau^2)^{\frac{1}{2}}]. \end{aligned} \tag{44}$$

$P(\tau, \psi)$, $Q(\tau, \psi)$ are defined in equations (9) and (10), and after rearrangement yield the following equations

$$P \cos \psi + Q \sin \psi = j(4\pi)^{-1}k_o^2 b \sum_{n=-\infty}^{+\infty} I_n J_n'(k_o b\tau) \exp \left[jn\left(\psi - \frac{\pi}{2}\right)\right], \tag{45}$$

$$P \sin \psi - Q \cos \psi = (4\pi\tau)^{-1} n k_o \sum_{n=-\infty}^{+\infty} I_n J_n(k_o b\tau) \exp \left[jn\left(\psi - \frac{\pi}{2}\right)\right], \tag{46}$$

where the prime denotes differentiation with respect to the argument. The ϕ component of the reflected field is contained in equation (29) and is explicitly given by

$$\begin{aligned} E_\phi^{(r)}(\rho, \phi, z) = \int_0^\infty \int_C E_{2\phi}(\tau, \psi) \\ \cdot \exp [jk_o\rho\tau \cos (\psi - \phi) - jk_o z(1 - \tau^2)^{\frac{1}{2}}] \, d\psi \tau \, d\tau. \end{aligned} \tag{47}$$

The contour of integration C in the complex ψ plane is to be chosen from physical considerations. An examination of equations (42), (45) and (46) shows that the ψ integrals $\mathcal{J}_1$, $\mathcal{J}_2$ are of the form

$$\mathcal{J}_{1,2} = \int_C {\sin \atop \cos} (\psi - \phi) \exp \left[jn\left(\psi - \frac{\pi}{2}\right) + jk_o\rho\tau \cos (\psi - \phi)\right] d\psi. \tag{48}$$

Since $\mathcal{J}_{1,2}$ are solutions of Helmholtz equation in cylindrical coordinate system, we expect them to be cylindrical functions (compare with Ref. 26, pp. 367–368). The requirement that they be bounded when the argument of the Bessel functions approach zero determines that they are the ordinary J functions. Thus we obtain,

$$\mathcal{J}_1 = 2\pi n(k_o\rho\tau)^{-1}J_n(k_o\rho\tau)\exp(jn\phi), \tag{49}$$

$$\mathcal{J}_2 = -j2\pi J_n'(k_o\rho\tau)\exp(jn\phi). \tag{50}$$

Substitution of equations (42), (45), (46), (48), (49) and (50) into equation (47) yields the following expression for the ϕ component of the reflected field

$$E_\phi^{(r)}(\rho, \phi, z) = 2\pi \sum_{n=-\infty}^{+\infty} I_n z_n(\rho, z)\exp(jn\phi), \tag{51}$$

where

$$\begin{aligned} z_n(\rho, z) = (4\pi)^{-1}k_o^2 b\eta_o \int_0^\infty & [n^2(k_o^2\rho b)^{-1}J_n(k_o\rho\tau)J_n(k_o b\tau)\tau^{-1}(1-\tau^2)^{\frac{1}{2}}R_\parallel(\tau) \\ & - J_n'(k_o b\tau)J_n'(k_o\rho\tau)\tau(1-\tau^2)^{-\frac{1}{2}}R_\perp(\tau)] \\ & \cdot\exp[-jk_o(z+d)(1-\tau^2)^{\frac{1}{2}}]\,d\tau. \end{aligned} \tag{52}$$

In particular,

$$\begin{aligned} z_n(b, d) = (4\pi)^{-1}k_o^2 b\eta_o \int_0^\infty & [n^2(k_o b)^{-2}J_n^2(k_o b\tau)\tau^{-1}(1-\tau^2)^{\frac{1}{2}}R_\parallel(\tau) \\ & - J_n'^2(k_o b\tau)\tau(1-\tau^2)^{-\frac{1}{2}}R_\perp(\tau)]\exp[-j2k_o d(1-\tau^2)^{\frac{1}{2}}]\,d\tau. \end{aligned} \tag{53}$$

IV. CONDUCTING CIRCULAR LOOP OVER A SEMI-INFINITE MEDIUM

In this section we consider a perfectly conducting thin circular loop of mean radius b situated over the semi-infinite medium (Fig. 1) with its plane parallel to the interface. The radius of the wire forming the loop is a and the height of the loop above the interface is d.

4.1 *Current Distribution*

Let $E_\phi^{(o)}(\phi)$ be the ϕ component of the applied electric field, $I^{(o)}(\phi)$ the current distribution that would be created by $E_\phi^{(o)}(\phi)$ on the loop if it were situated in free space and $I(\phi)$ the current distribution caused by $E_\phi^{(o)}(\phi)$ when the loop is situated as shown in Fig. 1. Typical examples

of $E_\phi^{(o)}(\phi)$ are the slice generator (used in admittance computations) and the ϕ component of the electric field that would exist at the loop location in the absence of the loop (as in scattering problems).

The relation between the applied tangential electric field and the currents induced on a loop takes the form of coupled integral equations which are extremely difficult to solve.[6] However if the loop is "thin" ($a \ll b$, $a \ll \lambda$) the current distributions on the loop is given accurately by the so-called "one-dimensional integral equation." Thus we have

$$E_\phi^{(o)}(\phi) = \int_0^{2\pi} G(\phi - \phi')I^{(o)}(\phi')\, d\phi', \tag{54}$$

where

$$G(x) = j(4\pi)^{-1}\eta_o\left[k_o b \cos x + (k_o b)^{-1}\frac{\partial^2}{\partial x^2}\right](2\pi)^{-1} \cdot \int_{-\pi}^{+\pi} \{R(x) \exp[-jk_o R(x)]\}_{a \to 2a \sin\theta}\, d\theta, \tag{55}$$

$$R(x) = b[4\sin^2(x/2) + (a^2/b^2)]^{\frac{1}{2}}. \tag{56}$$

A formal solution to equation (54) is obtained by using Fourier series representations as follows. Let

$$I^{(o)}(\phi) = \sum_{n=-\infty}^{+\infty} I_n^{(o)} \exp(jn\phi), \tag{57}$$

$$E_\phi^{(o)}(\phi) = \sum_{n=-\infty}^{+\infty} \alpha_n^{(o)} \exp(jn\phi), \tag{58}$$

$$G(\phi - \phi') = \sum_{n=-\infty}^{+\infty} \beta_n \exp[jn(\phi - \phi')], \tag{59}$$

where

$$I_n^{(o)} = (2\pi)^{-1}\int_0^{2\pi} I(\phi)\exp(-jn\phi)\, d\phi, \tag{60}$$

$$\alpha_n^{(o)} = (2\pi)^{-1}\int_0^{2\pi} E_\phi^{(o)}(\phi)\exp(-jn\phi)\, d\phi, \tag{61}$$

$$\beta_n = (2\pi)^{-1}\int_0^{2\pi} G(x)\exp(-jnx)\, dx. \tag{62}$$

Substitution of equation (57), (58) and (59) into equation (54) yields the following relation between the Fourier coefficients

$$I_n^{(o)} = (2\pi)^{-1}(\alpha_n^{(o)}/\beta_n). \tag{63}$$

The current distribution is given by

$$I^{(o)}(\phi) = (2\pi)^{-1} \sum_{n=-\infty}^{+\infty} (\alpha_n^{(o)}/\beta_n) \exp(jn\phi). \tag{64}$$

Let $E_\phi^{(1)}(\phi)$ be the tangential electric field, at the loop, of the reflected field whose incident field is caused by $I^{(o)}(\phi)$ and let $I^{(1)}(\phi)$ be the current distribution on the loop caused by $E_\phi^{(1)}(\phi)$. Let

$$E_\phi^{(1)}(\phi) = \sum_{n=-\infty}^{+\infty} \alpha_n^{(1)} \exp(jn\phi), \tag{65}$$

$$I^{(1)}(\phi) = \sum_{n=-\infty}^{+\infty} I_n^{(1)} \exp(jn\phi). \tag{66}$$

Then

$$I_n^{(1)} = (2\pi)^{-1}[\alpha_n^{(1)}/\beta_n] \tag{67}$$

[compare with equation (63)]. Also from equations (51) and (65) we obtain

$$\alpha_n^{(1)} = 2\pi I_n^{(o)} z_n(b, d). \tag{68}$$

Hence

$$I_n^{(1)} = I_n^{(o)}[z_n(b, d)/\beta_n]. \tag{69}$$

In general, if $I_n^{(k)}$ denotes the nth Fourier coefficient of current distribution $I^{(k)}(\phi)$, induced by the kth reflected field we have

$$I_n^{(k)} = I_n^{(k-1)}[z_n(b, d)/\beta_n], \qquad k = 1, 2, \cdots . \tag{70}$$

Let

$$I(\phi) = \sum_{n=-\infty}^{+\infty} I_n \exp(jn\phi). \tag{71}$$

Then

$$I_n = \sum_{k=0}^{\infty} I_n^{(k)}, \tag{72}$$

that is,

$$\begin{aligned} I_n &= I_n^{(o)} + I_n^{(1)} + I_n^{(2)} + \cdots, \\ &= I_n^{(o)}\{1 + [z_n(b, d)/\beta_n] + [z_n(b, d)/\beta_n]^2 + \cdots\}, \\ &= I_n^{(o)}\{1 - [z_n(b, d)/\beta_n]\}^{-1}, \quad \text{provided} \quad |[z_n(b, d)/\beta_n]| < 1. \end{aligned} \tag{73}$$

Henceforth, we assume that $|z_n(b, d)/\beta_n| < 1$. Substituting for $I_n^{(o)}$

in equation (73) from equation (63) we obtain

$$I_n = (2\pi)^{-1}\alpha_n^{(o)}[\beta_n - z_n(b, d)]^{-1}, \tag{74}$$

$$I(\phi) = (2\pi)^{-1} \sum_{n=-\infty}^{+\infty} \alpha_n^{(o)}[\beta_n - z_n(b, d)]^{-1} \exp jn\phi. \tag{75}$$

The following expression for β_n is obtained by simplifying equation (62):

$$\beta_n = j(4\pi k_o b^2)^{-1}\eta_o(2/\pi) \cdot \int_0^{\pi/2} \{\tfrac{1}{2}(k_o b)^2[M_{n+1}(a) + M_{n-1}(a)] - n^2 M_n(a)\}_{a\to 2a\sin\theta}\, d\theta, \tag{76}$$

where

$$M_n(x) = \frac{1}{\pi}\int_0^{\pi/2} (\sin^2\theta + x^2/4b^2)^{-\frac{1}{2}} \cdot \exp[-j2k_o b(\sin^2\theta + x^2/4b^2)^{\frac{1}{2}}]\cos(2n\theta)\, d\theta. \tag{77}$$

Let

$$K_n = (2/\pi)\int_0^{\pi/2} M_n(a)\Big|_{a\to 2a\sin\theta} d\theta. \tag{78}$$

Then

$$\beta_n = j(4\pi k_o b^2)^{-1}\eta_o[\tfrac{1}{2}(k_o b)^2(K_{n+1} + K_{n-1}) - n^2 K_n]. \tag{79}$$

4.2 *Input Admittance*

Let the primary source be a slice generator (delta function source) of voltage V located at $\phi = 0$. That is

$$E_\phi^{(o)}(\phi) = [V\delta(\phi)/b]. \tag{80}$$

Substituting equation (80) into equation (61), we obtain

$$\alpha_n^{(o)} = (V/2\pi b). \tag{81}$$

Substitution of equation (81) into equation (75) yields

$$I(\phi) = V(4\pi^2 b)^{-1} \sum_{n=-\infty}^{+\infty} [\beta_n - z_n(b, d)]^{-1} \exp(jn\phi). \tag{82}$$

The admittance Y at the input terminals at $\phi = 0$ is given by

$$Y = I(0)/V = (4\pi^2 b)^{-1} \sum_{n=-\infty}^{+\infty} [\beta_n - z_n(b, d)]^{-1}. \tag{83}$$

The use of the delta function generator will give rise to an infinite input admittance[6] so that the series in equation (83) is divergent. However this difficulty is overcome by computing the difference between the admittances of two loops of different radii.[27]

V. SPECIAL CASES

5.1 *The Magnetic Dipole Over a Semi-Infinite Medium*

When the radius, b, of the loop becomes very small compared to the wavelength (that is, $k_o b \ll 1$) the current distribution on the loop becomes uniform. This enables us to retain only the zeroth terms in the infinite series representing the different quantities of interest. The field of a dipole over ground is well discussed in the literature and will not be considered here. The input impedance of a dipole over a semi-infinite medium is of considerable interest and may be obtained from equation (83). Thus we obtain

$$Z_{in} = 4\pi^2 b[\beta_o - z_o(b, d)]. \tag{84}$$

The term $4\pi^2 b\beta_o$ represents the input impedance of the loop in the absence of the semi-infinite medium and the term $-4\pi^2 bz_o$ represents the contribution of the semi-infinite medium. That is

$$Z_{in} = Z_{pri} + Z_{sec} , \tag{85}$$

where

$$Z_{pri} = 4\pi^2 b\beta_o , \tag{86}$$

$$\begin{aligned} Z_{sec} &= -4\pi^2 bz_o(b, d) \\ &= \pi(k_o b)^2 \eta_o \int_0^\infty J_1^2(k_o b\tau)\tau(1 - \tau^2)^{-\frac{1}{2}} R_\perp(\tau) \\ &\qquad \cdot \exp\,[-j2k_o d(1 - \tau^2)^{\frac{1}{2}}]\, d\tau. \end{aligned} \tag{87}$$

5.2 *Thin Circular Loop Over a Perfectly Conducting Plane*

Let

$$z_n^*(b, d) = \lim_{\sigma\to\infty} z_n(b, d). \tag{88}$$

When $\sigma \to \infty$ the reflection coefficients simplify to

$$R_\parallel(\tau) = +1, \qquad R_\perp(\tau) = -1. \tag{89}$$

Substituting equation (89) into equation (53), we obtain

$$z_n^*(b, d) = (4\pi)^{-1}k_o^2 b\eta_o$$
$$\cdot \int_0^\infty [n^2(k_o b)^{-2}J_n^2(k_o b\tau)\tau^{-1}(1 - \tau^2)^{\frac{1}{2}} + J_n'^2(k_o b\tau)\tau(1 - \tau^2)^{-\frac{1}{2}}]$$
$$\cdot \exp [-j2k_o d(1 - \tau^2)^{\frac{1}{2}}] \, d\tau. \quad (90)$$

The integral in equation (90) may be simplified, by making use of the properties of Bessel functions, and yields

$$z_n^*(b, d) = (4\pi b)^{-1}\eta_o[\tfrac{1}{2}(k_o b)^2(\zeta_{n-1} + \zeta_{n+1}) - n^2\zeta_n], \quad (91)$$

where

$$\zeta_n = \int_0^\infty \tau(1 - \tau^2)^{-\frac{1}{2}}J_n^2(k_o b\tau) \exp [-j2k_o d(1 - \tau^2)^{\frac{1}{2}}] \, d\tau. \quad (92)$$

An alternate expression for ζ_n is obtained as follows:

$$J_n^2(k_o b\tau) = \frac{1}{\pi}\int_0^\pi J_o(2k_o b\tau \sin \theta) \cos (2n\theta) \, d\theta; \quad (93)$$

substituting equation (93) into equation (92) and changing the order of integration we obtain

$$\zeta_n = j(k_o b)^{-1}M_n(2d), \quad (94)$$

where $M_n(x)$ is defined by equation (77). Therefore,

$$z_n^*(b, d) = j(4\pi k_o b^2)^{-1}$$
$$\cdot \eta_o\{(\tfrac{1}{2})(k_o b)^2[M_{n+1}(2d) + M_{n-1}(2d)] - n^2 M_n(2d)\}. \quad (95)$$

A comparison of equations (95) and (76) reveals a strong similarity between the expressions for β_n and $z_n^*(b, d)$. The input admittance of a thin circular loop over a perfect ground plane is given by

$$Y = (4\pi^2 b)^{-1} \sum_{n=-\infty}^{+\infty} [\beta_n - z_n^*(b, d)]^{-1}. \quad (96)$$

The above formula agrees, with that derived by Iizuka and others,[20] for the input admittance $Y^{(a)}$ of a loop in the presence of an identical coaxial loop carrying a current distribution which has an opposite phase. They use the simpler Kernel given by Storer[5] to computer β_n, a procedure satisfactory for small loops. However they make use of the similarity between β_n and $z_n^*(b, d)$ to compute the latter using the approximate expressions given by Storer. This procedure will yield erroneous results for large separation, d, for the following reason.

The approximate expressions for β_n given by Storer, or for that matter Wu, are valid for $k_o a \ll 1$. The corresponding condition to be

imposed in the evaluation of $z_n^*(b, d)$ is, $(2k_o d) \ll 1$. Thus it is seen that the approximate expressions of Storer give accurate values of $z_n^*(b, d)$ only for very small separations.

VI. NUMERICAL COMPUTATIONS

Equation (76) was further analyzed and approximate expressions for the β_n coefficients were derived in terms of Bessel and Legendre functions. The integral defining the z_n coefficients could not be expressed in terms of known functions and so was evaluated by numerical integration. However, in the evaluation of z_n^* it was possible to use some of the formulae developed for β_n for small values of d [compare with equation (95)]. The numerical integration was carried out by using the Romberg integration scheme. All the computations were done by FORTRAN programs on a GE-635 computer.

Figure 4 shows the variation of the input admittance of a loop, when it is in free space, over moist earth and over an infinitely con-

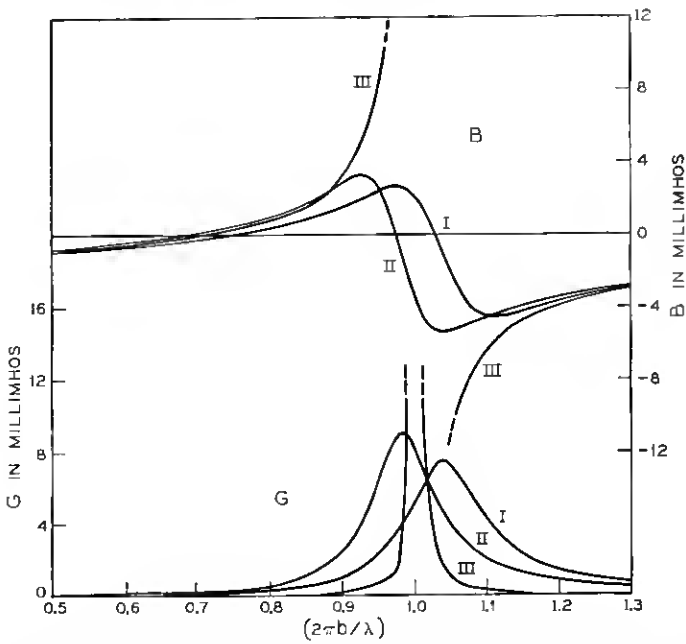

Fig. 4—Input admittance of a circular loop, over different media, as a function of frequency ($d/b = 0.25$; $a/b = 0.002$). G—conductance; B—susceptance. I—loop in free space; II—loop over moist earth; III—loop over perfect ground ($\sigma \to \infty$).

ductive ground plane, as a function of frequency. The values of the various parameters used in these calculations are $2\pi b = 30$ meters, $f = 5$ MHz to 13 MHz, $d/b = 0.25$, $a/2b = 0.001$, $\sigma = 5$ millimhos/meter, and $\epsilon/\epsilon_0 = 15$. The last two parameters characterize the moist earth. The frequency range was deliberately chosen so that the moist earth cannot be approximated either as a highly conductive medium (low frequency approximation) or as a lossless dielectric (high frequency approximation).

The real part G of the input admittance shows its characteristic peak near $k_o b = 1$ (these occur for values of $k_o b$ near 1, 2, 3, $\cdots$) but the exact location of the peak as well as its magnitude depends on the medium below the loop. When the loop is located above a highly conducting ground plane the resonance is particularly sharp at $k_o b = 1$ since at this frequency the loop and its image are exactly half wavelength apart. The imaginary part B of the input admittance changes from inductive to capacitive near $k_o b = 0.7$ and back to inductive near $k_o b = 1$. Here again the transition at $k_o b = 1$ for the loop over a highly conducting ground plane is almost discontinuous.

Figure 5 shows the variation of the input admittance of a loop over a highly conductive ground plane as a function of the distance between the loop and the ground plane. The curves are plotted for $k_o b = 1$, $a/2b = 0.001$ and d/b ranging from 0.25 to 5.0. It is observed that as d/b increases, the input admittance approaches the free space value in an oscillatory manner.

The aforementioned calculations are presented only as examples of the different types of investigations that may be carried out based on the theory developed. The computer programs developed in this connection are very general and may be used for computations of loops as large as $k_o b = 10$.

VII. SUMMARY

The problem of a thin, perfectly conducting, circular loop situated in free space over a semi-infinite homogeneous isotropic medium was solved. Expressions for the current distribution on the loop caused by an arbitrary time harmonic source [equation (75)] and the input admittance [equation (83)] were derived. The results are applied to special cases to evaluate the input impedance of a vertical magnetic dipole over a semi-infinite medium [equation (84)] and the input admittance of a circular loop over a perfectly conducting ground plane [equation (96)]. Some numerical results are also given. The analysis for the general

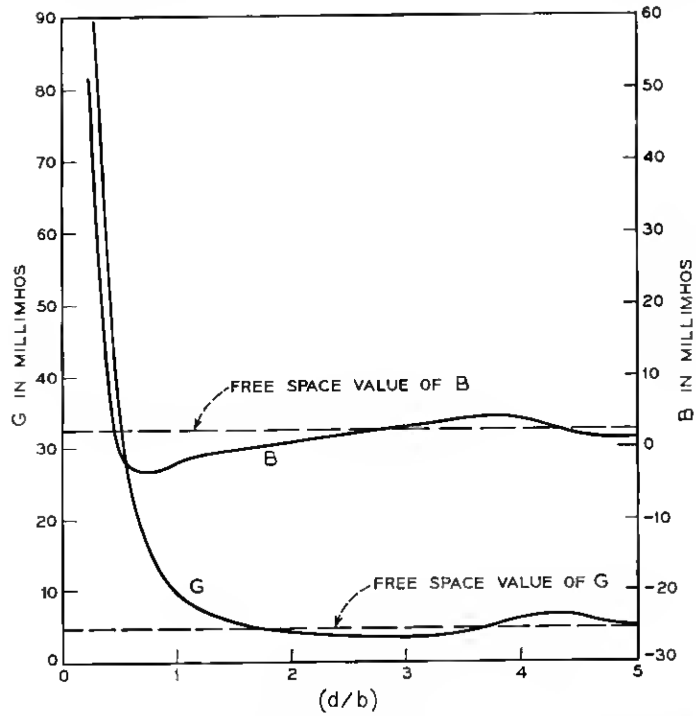

Fig. 5—Input admittance of a circular loop over an infinitely conducting ground plane as a function of height ($k_0b = 1$, $a/b = 0.002$). G—conductance; B—susceptance.

case of a circular loop in an arbitrary homogeneous medium (as opposed to free space assumed here) over another homogeneous medium can be easily done by modifying the parameters.

VIII. ACKNOWLEDGMENTS

The author wishes to thank Messrs. D. A. Alsberg, J. W. Carlin, G. T. Hawley, and R. D. Tuminaro for encouragement and numerous helpful comments. Computational assistance provided by S. Renna is gratefully acknowledged.

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